Bond percolation in a square lattice in presence of a 'magnetic field'†

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Abstract. We present a calculation of the bond percolation problem in a square lattice in presence of a 'magnetic field', using the position space renormalisation group and cells of dimension $b \times b$, where b runs from 2 up to 5. Due to symmetry, the calculation splits into two parts, one determining the 'thermal' exponent ν and the other, the 'magnetic' exponent η. For the largest cell in each case, we get $\nu = 1\cdot355$ ($b = 5$) and $\eta = 0\cdot244$ ($b = 4$), in good agreement with series results of Dunn *et al*. Comments are made on the extrapolation of the results to $b = \infty$.

1. Introduction

We have calculated the critical exponents for the bond percolation problem in a square lattice in presence of a 'magnetic field'. The 'magnetic field' is simulated by the introduction of a 'ghost' site, connected to every lattice site by 'ghost' bonds having probability h of being active (Kasteleyn and Fortuin 1969, Reynolds *et al* 1977, Marland and Stinchcombe 1977).

We use the position space renormalisation group (PSRG) (Young and Stinchcombe 1975, Reynolds *et al* 1977). First, we recall some features of PSRG regarding its application to percolation. Consider for simplicity a finite $b \times b$ square lattice with its $N = 2b^2$ nearest-neighbour bonds (figure 1), each having probability p of being active. One can count, out of the 2^N possible configurations, those in which it is possible to go
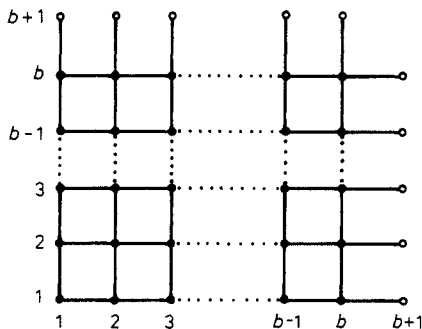

Figure 1. A finite $b \times b$ square lattice; note that the horizontal bonds in row $b + 1$ are not present; analogously for column $b + 1$; white sites are empty. In this way, such a cell can reproduce the infinite square lattice, through proper translations.

from row (or column) 1 to row (or column) $b + 1$ by passing only through active bonds. Such configurations will be called 'percolating'; this is our definition of percolation. Of course, the probability that such a lattice 'percolates' is given by

$$P_b(p) = \sum p^L(1-p)^{N-L} \tag{1}$$

where L is the number of active bonds in a given percolating configuration, and the sum extends over all such configurations. Our definition of $P_b(p)$ is the same as the definition of the probability $S_n(p)$ of Seymour and Welsh (1978) for the 'sponge percolation'
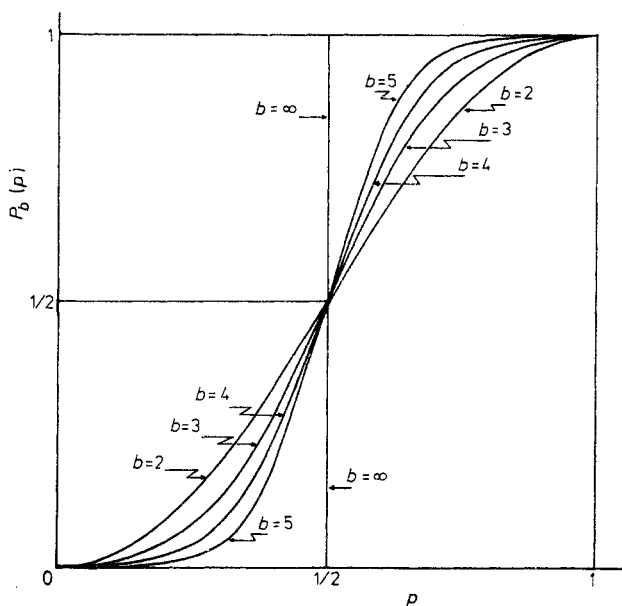

Figure 2. A plot of $P_b(p)$ for finite values of b, and the extrapolation for $b \to \infty$.

problem. The behaviour of $P_b(p)$ is plotted in figure 2, where it is also shown that for $b \to \infty$ we expect to have

$$P_\infty(p) = \begin{cases} 0 & p < \frac{1}{2} \\ 1 & p > \frac{1}{2} \end{cases},$$

according to the accepted result that the critical probability for bond percolation in a square lattice is $\frac{1}{2}$. See Seymour and Welsh (1978), Russo (1978) for a detailed discussion of this point. All curves cross at the point $p = \frac{1}{2}$ (see § 2 for comments). The PSRG procedures overcome the practical impossibility of exactly calculating $P_b(p)$ for large values of b by imagining, say, a $b^2 \times b^2$ square lattice, built up by cells of size $b \times b$, and approximating the 'percolation' probability $P_{b^2}(p)$ by the iterated probability $P_b[P_b(p)]$, where $P_b(p)$ is the exact probability of getting across a cell, and approximates the probability p' of having two nearest-neighbour sites connected in the renormalised lattice (see figure 3). As the iterations are repeated, one expects to simulate the behaviour of the true infinite lattice.

In the present case, there are two probabilities, p and h, to be considered; hence, we shall have two recursion relations. If renormalised probabilities are denoted by a
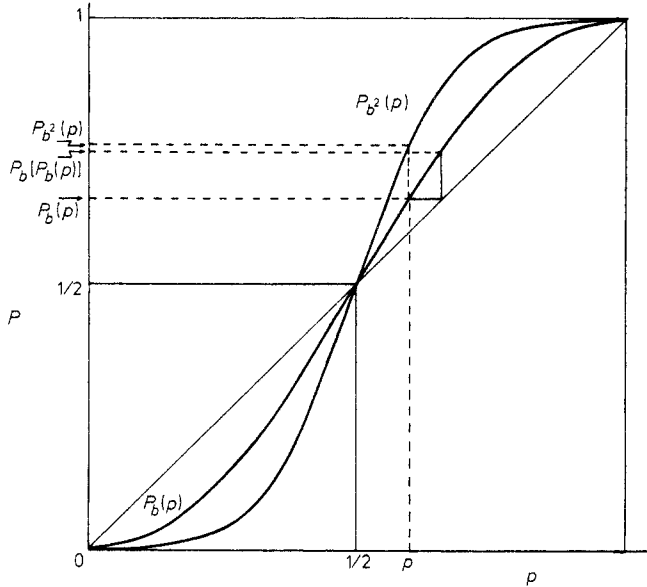
Figure 3. A schematic plot of the PSRG procedure. See text for further details.

prime, one has, after one iteration and for a given b (see figures 4 and 5),

$$p' = P_b(p, h) \tag{2a}$$

$$h' = H_b(p, h) = hR_b(p, h) \tag{2b}$$

where h is the probability of a 'ghost' bond being active; $P_b(p, h)$ is the probability of 'getting across' the cell, including also all possible paths through the ghost site. $H_b(p, h)$ is the probability of reaching the ghost site starting from given points in the original cell (see § 3). In (2b) we have factored out h; this clearly is always possible.

From (2) it follows that the fixed point is $p = p^*$ and $h = h^* = 0$, where p^* is the fixed point of the corresponding recursion relation in absence of the magnetic field (Niemeyer and Van Leeuwen 1974). In other words, p^* is the fixed point of

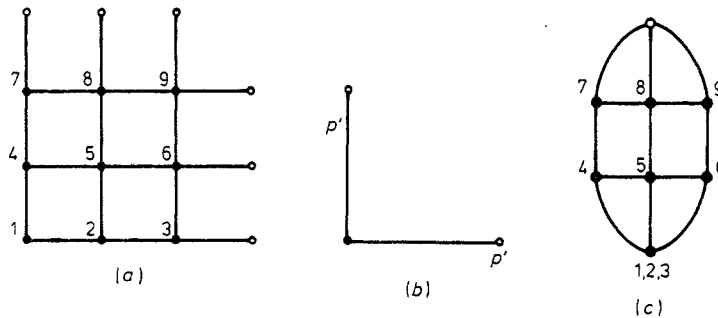$$p' = P_b(p, 0) \equiv \tilde{P}_b(p). \tag{3}$$

Figure 4. (a) A 3×3 cell in the original lattice; each bond has probability p of being active. It is transformed by PSRG into the cell in (b), where now each bond has probability p' of being active. Note that both cells reproduce the lattice by suitable translations. Following Tsallis (1978) we collapse the 'entries' and 'exits' of the cell and eliminate the external lateral bonds, since they do not contribute for vertical percolation, thus obtaining the diagram in (c). This one is further reduced to series and parallel combinations.
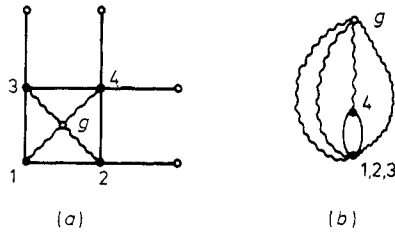
Figure 5. (*a*) A 2×2 cell with a ghost site (*g*). The wavy lines represent the ghost bonds, with a probability *h* associated to each. *h'* is the renormalised probability of reaching the ghost site. Taking points 1, 2 and 3 as entries and using the procedure described in figure 4, we arrive at the diagram shown in (*b*).

Now, consider the matrix A which has as its elements $a_{11} = \partial p'/\partial p$, $a_{12} = \partial p'/\partial h$, $a_{21} = \partial h'/\partial p$ and $a_{22} = \partial h'/\partial h$, where the derivatives are evaluated at the fixed point. As $a_{21} = 0$, it follows that the 'thermal' eigenvalue is $\lambda_1 = a_{11}$, and the 'magnetic' eigenvalue is $\lambda_2 = a_{22}$.

The exponents ν and η (Reynolds *et al* 1977) are then given by

$$\nu = \frac{\ln b}{\ln \lambda_1} \qquad \text{and} \qquad \eta = d + 2 - \frac{2 \ln \lambda_2}{\ln b}$$

where $d = 2$ is the space dimensionality. From these expressions, together with the scaling relations, all other exponents can be obtained.

2. The exponent ν

The above discussion shows that the determination of the exponent ν is independent of the presence of a 'magnetic field', as it should be; we need only consider the recursion relation (3). We have worked out this relation exactly for $b = 2$, 3, 4 and 5. These recursion relations are generated through a computer program and make use of the simplifying procedures introduced by Tsallis (1978) for reducing the cells to series and parallel combinations.

Following the definition of percolation stated in § 1, we consider for p', in the case $b = 3$ (see figure 4), all paths beginning at points 1, 2 or 3 and leaving the cell through a vertical bond which begins at points 7, 8 or 9; and similarly for other values of b. It should be noticed that in more complicated cases, such as the 'magnetic' case (see § 3), or if there are both first- and second-neighbour bonds, many definitions of percolation might at first sight seem reasonable, and in general give quite different results for the critical exponents. This question will be discussed with further details in a future paper on a more general bond percolation problem.

The degrees and numerical coefficients of the polynomials increase rapidly with b; for $b = 5$, the recursion relation is of degree 41. For $b = 2$, for instance,

$$p' = 2p^2 + 2p^3 - 5p^4 + 2p^5.$$

The critical probability is $p_c = p^* = \frac{1}{2}$, for all polynomials we have calculated. In fact, it is always $\frac{1}{2}$ for any b, provided the cell has the appropriate symmetry as in figure 1. This can be seen from the following reasoning. The recursion relation, for a given b, has the

form

$$p' = \sum_n C_n p^n q^{N-n} \tag{4}$$

with $q = 1 - p$. C_n is the number of configurations of n active bonds such that the cell can be traversed in a given direction, say, vertically. The sum in (4) runs from the minimum number of active bonds needed for traversing the cell—which is clearly equal to b—up to the total number $N = 2b^2$ of bonds in the cell. Now we state, using symmetry considerations, that the probability $q' = 1 - p'$ that the cell does not percolate is given by (4), exchanging p and q, i.e.

$$q' = \sum_n C_n q^n p^{N-n} \tag{5}$$

with the *same* coefficients C_n. These coefficients are now interpreted as giving the number of configurations of n *non-active* bonds such that we do *not* succeed in traversing the cell. This can be seen by relating each configuration of the bonds in the cell to a complementary configuration in the dual cell, built up by drawing a bond crossing each one on the original cell. Each bond in the dual cell is considered as non-active if the crossed bond on the original cell is active, and vice versa. Thus, to each vertical percolating configuration in the cell corresponds a horizontal non-percolating configuration in the dual cell. As the cell in figure 1 is self-dual for any value of b, equality of the coefficients C_n follows immediately.

Clearly, from (4) and (5) we can conclude that if $p = q$ we shall have $p' = q'$, hence $p_c = p^* = \frac{1}{2}$ is a fixed point (Sykes and Essam 1963, 1964). See also Seymour and Welsh (1978).

The calculated values of ν for each b are displayed in table 1. The reader may verify that the results show a fairly rapid convergence towards the accepted value $\nu = 1\cdot34 \pm 0\cdot02$ (Dunn *et al* 1975).

Table 1. Calculated values of the exponents ν and η for several values of b.

b	ν	η
2	1·428	0·186
3	1·380	0·228
4	1·363	0·244
5	1·355	—
∞ (*)	1·341	0·301

(*) extrapolated (see text).

3. The exponent η

Having established that $(2a)$ always allows $p^* = \frac{1}{2}$ as a fixed point for our diagrams, it is clear that we need not work out explicitly equation $(2a)$ as far as the exponent η is concerned. Also, due to the fact that $h^* = 0$, all we need is $(2b)$ to first order in h. So we rewrite $(2b)$ as

$$h' = h R_b^0(p) + O(h^2)$$

and then $\lambda_2 = R_b^0(1/2)$.

In order to calculate h', the renormalised probability of reaching the 'ghost' site, we take points 1, 2 or 3 as entry points in the case $b = 2$ (see figure 5) and analogously for larger b. That is, we take into account all paths arriving at the 'ghost' site which begin at any point on the external boundaries of the cell. Had we taken only points 1 and 2 as entry points, we would have got rather different values for the exponent η. In the case considered in § 2, it makes sense to consider only points 1 and 2 of figure 5 as entries, since we would then be concerned with 'vertical' percolation. However, for h' this distinction does not apply, and we must consider all paths, coming from below or from the left, such that they reach the 'ghost' site.

We have worked out the recursion relations $R_b^0(p)$ for $b = 2, 3$, and 4. For $b = 2$, for instance,

$$R_2^0(p) = 3 + 2p - p^2.$$

The calculated values for the exponent η are displayed in table 1.

4. Discussion

If we extrapolate our results to $b \to \infty$, by plotting ν against $1/b^2$, we get $\nu_\infty = 1 \cdot 341$.

Although this extrapolation is not quite rigorous in our case, since our values of b might be far from the asymptotic region, from figure 1 of Reynolds *et al* (1978) it can be seen that a plot of $\ln \lambda_1$ against $\ln b$ is a straight line down to small values of b. Our best plot was, instead, of ν against $1/b^2$ and a least-squares fit gave a straight line to one part in 10^5. Note that $2b^2$ is the number of bonds in the $b \times b$ cell.

For the exponent η the situation is more complicated. The value of η is somewhat uncertain and runs from $\eta = 0 \cdot 224$ to $\eta = 0 \cdot 187$ (Dunn *et al* 1975, Sykes *et al* 1976, Reynolds *et al* 1978, Dasgupta 1976). Our results for η (for $b \geq 3$) are higher than these and seem to be increasing with b, although it is not certain that we can extrapolate our results to $b \to \infty$ in this case.

Using the series values $\beta = 0 \cdot 138$ (Sykes *et al* 1976), $\nu = 1 \cdot 34$ (Dunn *et al* 1975) and the scaling relations, we find $\eta = 0 \cdot 206$ and then $y_2 = 1 \cdot 897$, where $\lambda_2 = b^{y_2}$. If we extrapolate our y_2 by plotting $\ln \lambda_2$ against $\ln b$, as in Reynolds *et al* (1978), the slope gives $y_2 = 1 \cdot 850$, which differs by only $2 \cdot 5\%$ from the above result. Nevertheless, the exponent η is very sensitive to this small variation in y_2 and for that reason, in our opinion, the question of calculating magnetic exponents through the position space renormalisation techniques deserves more attention.

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